1 General Set Up

1. Introduction: (5 mins):

   - 3 man seminar
   - personal motivation: spineless 4 manifolds
   - HFH mentioned around department, MSRI floer homotopy workshop - really hawt. So good to give a very basic introduction to set some context.
   - Budgeted 2 talks: no idea how long it'll go on for but here is the very crude plan: today we'll give some background and context for HFH
   - understanding all of the moving parts versus understanding its place in the ambient mathematical landscape. Illuminating

2. Hot and fast description of HFH: (5 mins)

   - package of 3 manifold invariants that have trickled down from symplectic topology and the study of 4 manifolds. For an oriented, closed 3 manifold Y, we denote them by

     $$\hat{HF}(Y), HF^{\pm}(Y), HF^{\infty}, HF^{o}$$

     In order to do so we need to define a chain complex $CF(Y)$, and a differential ∂ so that $\partial^2 = 0$, and we need this to be an invariant of Y up to chain homotopy.
   - Combinatorial in nature, which means good for knots! Analogue of this which is a strong invariant in the sense that the knot floer homology of a knot sitting inside S^3 detects its genus, and in fact if you look at the graded euler characteristic (the homology is bigraded), this is in fact the alexander polynomial.
   - they're very constructive which is nice!

3. resources (2mins): intro to HFH (2006), Jen Hom lecture series, Oszvath Simmons centre talk

4. Setting the scene: (12 mins)

   - 80s revolution in 4 manifold study: Donaldson ('83) Idea: study the solution space of a pde on X^4: the anti-self dual yang-mills equation.

     $$(X^4, g) \longrightarrow \mu(\phi) \longrightarrow \#\mu(\phi) \mod 2$$

   - seiberg-witten equations - allowed for this moduli space to be compact for free, and so you don't have to go through the process of compactification — get a numba
   - able to get a topological invariant which is somehow madness
   - want to be able to kind of do stuff with them - low dimensional topology is very constructive, and so we'd like to kind of be able to have some relationship between the invariants for manifolds when we glue them together.
   - the idea then is to ask whether we can assign some invariants to a 3-manifold bounding a 4 manifold, so that when we glue two 4 manifolds together along that boundary we recover the 4-manifold invariant. This was a question asked by Floer, along with the work of Atiyah, Taubes.

- This was really exciting stuff, proved stuff like there exist 4 manifolds that do not admit any smooth structures at all. Big M energy.
- Here comes HFH: oogalay to think about pdfs (sorry to my PDF queen), but if we're studying 3 manifold topology, then we better not be doing so by using something from the physics heavens up in 4 manifold lala land. At the very least, we should be able to approach it from the 3 manifold side and make sense of it.
- In pops Peter Oszvath and Zoltan Szabo, who tried to give the math world an answer to this by providing some combinatorial description of Seiberg-Witten homology. In some analogous fashion to Lagrangian floer homology. Somehow the 4 manifold picture would come back when we start thinking about spaces bounded by these 3 manifolds, in particular there's a lot of rich theory and properties that come out of it by thinking about homology cobordisms between 3 manifolds.

2 Lagrangian and Heegaard Floer Homology

5. Story of Lagrangian Floer homology,

- start with some symplectic manifold (X^{2n}, ω), and for a pair of Lagrangian submanifolds L_1, L_2 you can cook up some invariant of this pair.
- Roughly speaking, one can construct a chain complex from this symplectic manifold and Lagrangian submanifold pair by letting the chains be generated by (signed) intersection points of the submanifold
- easy enough, but beauty comes in how one defines the differential: Uses holomorphic disks.
- make mention of somehow with a choice of symplectic form you can get out an almost complex structure that is compatible with the form. I guess really I mean pseudo-holomorphic if I'm being pendantic, holomorphic with respect to this almost complex structure.
- maybe draw picture of it: manifold X, then Lagrangian submanifolds, then
- look at Whitney disks between two points, these form a group under juxtaposition and there's a natural multiplication of disks.
- space of holomorphic representatives of whitney disks between x and y: moduli space $\mathcal{M}(x, y)$.

  We saw that $Sym^g(\Sigma_g)$ has a complex structure induced from one on Σ_g, and T_α, T_β are totally real submanifolds of $Sym^g(\Sigma_g)$. Consider then the *moduli space* $\mathcal{M}(\phi)$ of holomorphic representatives of an element $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, i.e. the set of holomorphic representatives of the homotopy equivalence class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. In particular, a choice of almost complex structure on $\mathcal{M}(\phi)$ makes it into a manifold.

  And this allows us to define the differential operator:

$$\partial(x) = \sum_y \#(\hat{\mathcal{M}}(x, y))y$$

I've been told that this fact of $\partial^2 = 0$ is a manifestation of Gromov's compactness theorem for Whitney disks, but I have no further information on that.

6. Too much Floer, not enough Heegaard:

- This is really a general framework that we wanna use:
- $Y^3 \to$ some symplectic manifold $\to$ lagrangian submanifolds $\to$ Floer Homology
- Question is, how do you cook up an even dimensional manifold from a 3-manifold?
- Story turns to Heegaard, which gives a combinatorial decomposition of a 3-manifold into two pieces, each homotopic to a bouquet of circles.
- FIRST: talk about Heegaard splittings and how they're determined up to attaching circles
- splittings are not unique, but can be categorised by stabilisations. The attaching circles and the genus of the surface tell you all the information you'll ever need about the manifold topologically.
- standard position (there's a notion of how we orient these guys but I'm not going to bother about that)
- NEXT: how can we cook up a symplectic manifold from this? Answer: symmetric product.
- in fact a symplectic manifold and if you look at the image of the attaching circles under this construction, you get two symmetric tori which are Lagrangian submanifolds. We can then run this through the machine that is Lagrangian Floer homology (approximately) and get a Floer theory.
- SNEAKILY in the background we're also equipping the manifold with a $spin^c$ structure, which has some involvement in these Whitney disks but I don't really wanna go into that

Part II of Talk

1. Introduction:

    - Last time we spoke about the origins of HFH, and started working through its general construction.
    - Today: finish up the construction, maybe do an example of calculating the Heegaard Floer Homology of a space, and then talk about why we should maybe care about HFH some more.
    - Recall what Heegaard Floer homology does: it assigns to a 3 manifold (closed oriented, rational homology sphere) a package of topological invariants
    - How does it do this? Well it takes some 3 manifold, cooks up this weird symplectic manifold from it, and then feeds that data through a machine which we called Lagrangian Floer homology.
    - My plan: I'm going to spend literally like 5-10 mins recapping LFH so that we just remind ourselves of what we need to construct out of our 3 manifold to feed it through this machine. Once I've recapped that, we'll talk about how you can cook up a symplectic manifold from a 3 manifold.

2. Recap of LFH:

    - What does LFH do? Takes a symplectic manifold, with a pair of Lagrangian submanifolds, and from this it builds up a homology group.
    - How does it do this? Well, we like our pair of submanifolds to be transverse to each other - then for dimension reasons they must intersect in a bunch of points.
    - Draw picture in
    - This is the data that we want to capture in the homology theory. So let's let the chains be generated by these points.
    - In order to build one though, we need some relationship between these points.
    - I don't quite have a good intuition yet for why this is the relationship to think of between the points - the ideas behind this are motivated by Morse Homology I believe, as Floer theory is essentially an infinite dimensional version of it.
    - explain what a whitney disk is again. Say that you have up to homotopy $\pi_2(x, y)$, and then you look at one of those whitney disks: $\phi \in \pi_2(x, y)$. Then you look at the moduli space of holomorphic representatives of an element $\phi \in \pi_2(x, y)$. With a generic choice of almost complex structure, you can assemble the moduli space into a manifold, which you can quotient by a particular $\mathbb{R}$ action and that gives you a compact, 0-dimensional space.
    - gromov compacness thm for whitney disks means $\partial^2 = 0$.
    - So you input: (X, L_1, L_2), and output is some group $HF(L_1, L_2)$.
    - Right so this is the machine - we need the ingredients now! How do we go about that: Need to do two things: (1) figure out what makes a 3-manifold a 3-manifold, and (2) translate that data into a form that LFH likes to eat.

3. What makes a 3 manifold?

   - Handle decompositions of 3-manifolds. In the case of 3 manifolds though things become rather nice.
   - talk about handle decomposition of 3 manifolds, and how you can turn these diagrams upside down.
   - This leads us to think about heegaard diagrams.
   - In order to build the inner and outer structures, what we need to do is attach a 2 handle, and for that we need an attaching circle and a framing of the normal bundle.
   - This is all the data that we need to describe a 3 manifold. This is the data that we want to translate into a dough that we can run through the lagrangian floer homology machine.

4. Turning this data into a symplectic manifold. One of the potential miracles of symplectic geometry is that if you look at an oriented surface σ_g, and you take its g-fold symmetric product, then equipping this surface with an almost complex structure gives you an almost complex structure on the product space.

   > This complex manifold structure is defined using the fact that $S^d\mathbb{C}$ is homeomorphic to $\mathbb{C}^d$ via the first d elementary symmetric functions. However, if $\phi : \Sigma \to \Sigma$ is a smooth map, then $S^d\phi$ need not be smooth. (There is a way to get a smooth map on the symmetric products using the vortex equations, which was introduced in a paper of Salamon, but this smooth map is not the obvious set-theoretic map $S^d\phi$ considered above.) On the other hand, if ϕ is j-holomorphic, then $S^d\phi$ is smooth (indeed holomorphic) with respect to the smooth (complex) structure on $S^d\Sigma$ defined by j.

   This gives us our manifold. Why would we necessarily choose this guy? Well, this gives us a really nice environment for our attaching circles to present - it encodes them in a pair of lagrangian submanifolds.

   Look at T_α and T_β. These are in fact lagrangian submanifolds, and so this gives us the data that we need to run through our LFH machine.

5. For remainder of talk, we're going to go through a specific example calculating the Heegaard floer homology of the 3-sphere. Maybe this will shed some light on what exactly we mean by